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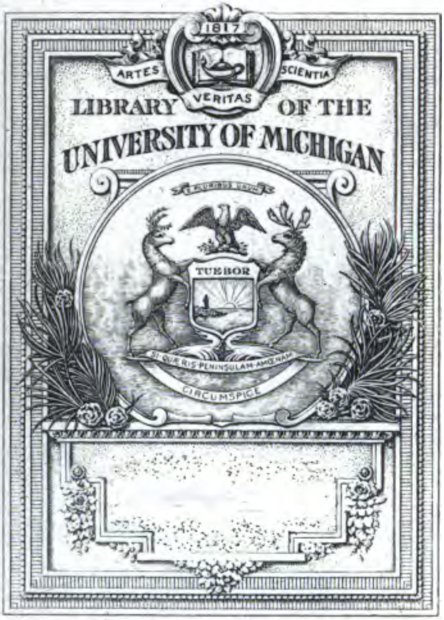
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GEOMETRY
IN THE GRAMMAR SCHOOL

An Essay

TOGETHER WITH ILLUSTRATIVE CLASS EXERCISES,
AND AN OUTLINE OF THE WORK FOR THE
LAST THREE YEARS OF THE
GRAMMAR SCHOOL

BY

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PREFACE.

THE public grammar schools educate about ninety-three per cent. of our population. The continued improvement of these schools is therefore a matter of great importance.

The college teacher of Education and Teaching, like the city superintendent, studies the workings of school systems, curricula, and principles of teaching, that he may help to mould public opinion, and contribute his share to the development of wise courses of study and good methods.

The writer of this little book therefore gladly availed himself of the opportunity to have a share in promoting the changes made during the present year in the work of the grammar schools of Cambridge. The book has grown out of lectures to the teachers of the seventh, eighth and ninth grades. Those lectures formed part of the plan whereby Harvard University gave instruction to teachers of the grammar schools in certain new subjects introduced into the curriculum.

The book comprises: —

A discussion of the subject-matter to be selected for grammar school work in geometry;

The method of teaching;

Illustrative class exercises;

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An outline of work in geometry for the last three years of the grammar school.

The aim and method of grammar school instruction in geometry, from the writer's point of view, are embodied in the summary of principles on pages 23–25, which the writer ventures to hope will receive careful consideration by all who may read this essay and the appended illustrative class exercises and outlines.

The assistance derived from Kehr's *Praktische Geometrie für Volks- und Fortbildungsschulen*, which has served as the basis of what is here presented, is cheerfully acknowledged.

PAUL H. HANUS.

HARVARD UNIVERSITY, March, 1893.

GEOMETRY IN THE GRAMMAR SCHOOL.

LIKE every other subject admitted into the curriculum, geometry has a value as knowledge and a value as discipline. Its value as knowledge is illustrated in all the manifold occupations of life; most of the trades and industries have a mathematical basis. Every worker in wood, tin, iron, stone — whether contractor, builder, tinsmith, mason, carpenter, or what not — must appeal to geometry to avoid waste of material and loss of time in useless experimenting; so also the farmer who wishes to know the extent of a piece of land in order to estimate the cost of reclaiming it or the amount of its possible productiveness, who has a drain or ditch to build, or a dam to construct, keenly feels the lack of such an elementary knowledge of geometry as would enable him to make simple preliminary measurements and save him both time and money. Further, "When it is considered that by geometry the architect constructs our buildings, the civil engineer our railways; that by a higher kind of geometry the surveyor makes a map of a county or of a kingdom; that a geometry still higher is the foundation of the noble science of the astronomer, who by it not only determines the diameter of the globe he lives upon, but as well the size of the sun, moon and planets, and their distances from us and from each other; when it is considered, also, that by this kind of geometry, with the assistance of a chart and a mariner's compass, the sailor navigates the ocean with success, and thus brings all nations into amicable intercourse—it will surely be allowed that its elements should be as accessible as possible."

Thus the knowledge value of geometry is obviously very great. Turn now to its disciplinary value. In its early lessons geometry develops correct notions of form and magnitude, and an appreciation of regularity and proportion: and through the definiteness and precision of its data it trains the mind to accuracy and clearness in grasping conditions. The way is thus easily and naturally prepared for a strictly scientific study of the subject later on, when through rigorous procedure from step to step by reference to fundamental principles, or previously established propositions, and the ruthlessness with which every error is detected and overthrown, it leads to the consciousness of full and solid achievement—the most powerful of all intellectual incentives to exertion. A firm grasp upon data, a rigorous exactness in reasoning, and the consequent achievement help to form the habit of insisting upon nothing less than adequate mastery, whatever the subject may be. This habit marks the trained mind.

We conclude therefore, that geometry has a place in the curriculum of the grammar school *because it yields a peculiar and important kind of knowledge and a highly desirable mental discipline.* This point having been decided, two important considerations present themselves: (I.), the selection of the subject-matter, and (II.), the method of teaching.

I. In the grammar school the knowledge value of a subject should never be subordinate to the disciplinary value. In the high school we may begin, and in the college we ought to begin to specialize. Not so in the grammar school. There the pupil is exploiting his capacity and his tastes, and is moreover acquiring the knowledge which he must have to rènder him useful, and to enable him to participate intelligently and appreciatively in common affairs. Accordingly we must decide at the outset what portions of geometry are to be selected for grammar school work, and what can be advantageously deferred. Two principles will guide us in making a choice of material from the vast store that the subject affords:

a. Grammar school instruction in geometry should give preference to those topics which have a practical application in the ordinary affairs of life.

b. In so doing special attention must be given to those propositions which can be established chiefly through observation, empirically: gradually the pupil must be led to undertake the easier deductive proofs.

In the "Outline of Work in Geometry for the last three years of the Grammar School," appended to this essay, these principles have been observed.

II. In the presentation of the subject the best results will be obtained only when the pupil has no text-book which contains the definitions and propositions. Let the teacher have *one* definite outline to follow (which may also be in the hands of the pupil), and let the teacher study one or more texts by all means; but let the pupil have no text except what he and the teacher make together. In this way the pupil learns *how to study* the subject, and difficulties vanish when the pupil really knows how to attack them. In this way, too, teacher and pupil *reconstruct the subject for themselves*, and in so doing realize the full value of every acquisition both in knowledge and power.

When geometry is not studied and taught as it should be, not only shall we fail to achieve the results at which we aim, but we may even produce results the reverse of those desired. Instead of interested attention, close application and accurate reasoning, we may develop carelessness, inattention, mind-wandering, and finally aversion and disgust. Nothing is more wearisome to the normal mind than the attempt to seize relations between data dimly perceived. The distress consequent upon being continually baffled is equalled only by the "exquisite misery of conscious weakness" which is sure to follow. When geometry is not properly studied and taught it is capable of producing just this weariness and the consequent discouraging sense of weakness. On the other hand, if undertaken in the right way it develops the power of independent effort and affords the just consciousness of strength that is born of achievement.

Of course geometry is not the only subject which may be so treated as to develop either weakness or strength. But the contrast between the good effect of a right method of dealing with geometry and the bad effect of a wrong one is particularly striking.

The peculiar simplicity of the fundamental principles of geometry when rightly presented, the ease with which its data may be grasped, and the certainty of its processes render it a peculiarly valuable instrument for intellectual training. There are few mental images so vivid as those representing geometrical concepts, and no logical processes affording greater definiteness and precision than the deductive proofs of geometrical reasoning. The vividness of geometrical images is greatest of course in good visualizers; but even poor visualizers will find geometrical forms the most vivid of their reproduced visual experiences. Parallel lines, vertical angles, cone, pyramid, sphere, — the objects themselves could hardly be more vivid than our mental representations of them, provided, of course, that the real things which these images represent have been fully and accurately perceived. This full and accurate perception of concrete magnitudes, as we shall see presently, is the basis of all real progress in geometry.

Again, two vertical angles are equal — the proof leaves no possible room for doubt. The area of a trapezoid is equal to the product of the altitude multiplied by half the sum of the parallel sides — every step rigorously proved, impossible to dissent from the conclusion. This is what we mean by mathematical certainty and precision. To render the attainment of such certainty and precision in deductive reasoning habitual is one aim of geometrical discipline.

It becomes evident then that the method of teaching geometry is of as great importance as the choice of the subject-matter. The evidences of a wrong method are not uncommon. When pupils of average ability and twelve or thirteen years of age or older, dislike geometry, and declare they never could do any thing with mathematics any way, it is usually because they have not studied the subject in the right way. One must of course admit great diversity in the depth of interest different pupils will develop in mathematics, and also great differences in the rate of progress that different pupils will make. But on the whole the belief that the ability to learn geometry implies any peculiar mental endowment, except industry and persistence, is erroneous. Successful teachers of geometry have sometimes avowed themselves believers in this doctrine, perhaps because they felt that they themselves were among the elect, and those who failed to achieve success in such instruction have sometimes maintained it because perhaps they could thus find consolation for conscious deficiencies in mathematical attainments and in teaching power. The doctrine itself, as usually understood, is, I believe, false. The man who is a producer in mathematics, who is a discoverer and extends the boundaries of the science — such a one has peculiar mental gifts, is a genius, born and not made. But just as there are thousands of persons who are capable of enjoying and appreciating the poem, or the work of art which they can not produce, so every normal mind can be led to understand, and, to some extent, to enjoy the discoveries and the creations of mathematical genius. "All mathematics depends upon simple concepts of space and number," Falk says. "He who cannot understand these is feeble-minded; the single concepts are combined by the simplest logical processes: he who cannot master them is incapable of reasoning, and will be equally unable to make any progress in other subjects." The simplicity and vividness of the fundamental concepts I have already pointed out. If another illustration were needed, reference need only be made to the Kindergarten, where the youngest children learn to grasp the significance of number and form with the help of geometrical figures.

The difficulty then which is experienced by some beginners in geometry, and the meagre achievements many pupils make, are not due either to the insuperable difficulties of the subject itself nor to the peculiar constitution of the learner's mind. The cause of the difficulty must be sought elsewhere.

I believe it is due to the method employed by both text-books and teachers. The old strictly deductive presentation of the subject cannot be successfully employed with beginners in geometry. Such a method will develop no interest in the subject, no zeal in its pursuit. How can the pupil develop interest in or profit by a course of mathematical reasoning the significance of which he cannot grasp, because the data are but dimly perceived and the purpose of which is but vaguely discerned, because it deals with a subject-matter lying, apparently, wholly beyond the range of his experience, and employing a curious phraseology which seems somehow of itself to bring the skilful manipulator out right. By hypothesis so and so: by construction, so and so: whence the conclusion, by the third axiom and Proposition XVIII. Intelligent pupils have committed demonstrations to memory, because, having failed to grasp the meaning, they could not divest themselves of a lingering suspicion that the demonstration depended upon a skilful use of the words themselves. When the pupil says "I don't know how it begins," he shows the same mental attitude that he presents to the acquisition of a mere verbal association, like learning a vocabulary or a tense-form in language work. Such a study of geometry must be unproductive. Dull eyes listlessly follow the teacher's illustrations and explanations. Indifference soon gives place to aversion, and ere long the pupil wonders how it is possible that any sensible person should be expected to work at such a thoroughly uninteresting subject. This is the nadir of failure. A subject which is capable of rousing the greatest interest has become an aversion. The pupil's attitude toward a whole range of intellectual activity has been fixed, perhaps permanently, and the (usually) false conviction of his own incapacity is gradually borne in upon him.

Such a failure, when not due to a lack of scholarship (a defect which on the whole is rather rare in teachers of geometry), is usually traceable, as I have said, to a wrong method. Teaching power is either a natural or an acquired capacity. Like every native power it can be greatly assisted by a study of principles. When teaching power has to be acquired, it is of course less effective, but derives *all* its efficacy from experience and from the study of principles and their application in the forms of presentation employed by others. Here then I purpose to enter upon a brief discussion of the general principles of method, for next to scholarship a teacher's power depends upon his method. I must recur to some principles of teaching, very familiar in theory, but of which we need to remind ourselves afresh whenever a new subject is undertaken, or an old subject is to be presented to a new class.

The principles of the teaching method are derived from psychology. Not from the psychology of books necessarily, and by no means exclusively, but from the study of mind, whatever may have been the means by which such study was facilitated and directed. Every skilful teacher works in accordance with psychological principles. To make him distinctly conscious of his principles and of their validity does not simply afford encouragement, but it must increase his efficiency. Having learned to observe how intelligence develops, how knowledge is acquired, and how character is evolved, the principles of a method of teaching which conform to these processes may be formulated. To enumerate all the principles of method would be to formulate directions for stimulating and guiding the mind's activity in accordance with every law afforded by psychology. In other words, he who possesses a knowledge of psychology at the same time possesses by implication the general principles of the teaching method. Such details must be treated singly and cannot receive attention here. We may however state a general principle which can be conveniently made to include all the others, as follows: —

The teaching method must enable the pupil to get, I., clear and definite individual notions,[1] and II., to pass from them to equally clear and definite general notions; to do this by the exercise of his own powers so that knowledge may be assimilated and power developed.[1]

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[1] An illustration will make clear the use of the terms "individual notion" and "general notion." Suppose the pupil is to learn that the sum

Now there are two ways of applying this principle in practice: 1st, by continuous exposition, 2d, by questioning. The first can be advantageously employed in the instruction of advanced youth and of adults, very rarely in teaching children, for which purpose the second way is to be employed as much as possible.

The teacher's skill in applying this principle in practice — i. e. the number of rational devices he will employ — depends upon his grasp of the logical structure of the subject and upon his power to study his own mind and the mind of his pupil. Knowing the logical structure of the subject he will present the steps of every part and all the parts in their proper sequence, and so as to bring out their proper relations. His ability to study mind will enable him to ascertain the mental pro-

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of the angles of a triangle is two right angles. Let him cut out, say four paper triangles — one equilateral, one isosceles, one scalene, and one right angled. Let him then take the first and cut off the angles. Let him then place these angles adjacent and with their vertices at a common point. The pupil now sees the fact in this particular instance, i. e. he has acquired the *individual notion : the sum of the angles of this triangle is two right angles.* If he goes through with the same process with each of the remaining triangles, his mind is asking about the sum of the angles of any triangle, i. e. he is prepared to receive the *general notion,* with its appropriate demonstration — *the sum of the angles of any triangle is two right angles.* Sometimes the demand for the general notion is not so easily developed. It depends usually upon the teacher's tact and skill however, rather than upon the particular topic.

[1] See W. Rein, *Pädagogik im Grundriss,* p. 111, Stuttgart, 1890.

cesses the pupil must undergo to master the subject, and to present the subject step by step in harmony with these processes.

I repeat, then, that the teacher's method[1] is as important as his knowledge of the subject matter.

Our general principal of method will be found to yield three special directions for instruction in geometry. The way to acquire clear and definite individual notions, and through them, clear and definite general notions, is to begin with a careful study of individual things and processes.

Hence, 1. *The first teaching of geometry must be object teaching.* Grammar school instruction in geometry cannot aim at general demonstrations, but it must enable the pupil to acquire the data and the simpler truths of geometry through experience—through measurement, drawing, construction, and superposition. While such acquisition of the individual notions of geometry is progressing, many general truths are slowly evolved, and before the end of the grammar school period the pupil knows much of his geometry as a science as well as a series of separate truths and processes. For some pupils this may not be the case. The time of transition from individual to general notions may not have come for them. But they, too, have acquired valuable knowledge and important data without which even when more mature the desired progress from individual to general notions would be impossible. "Without observation valid thinking is impossible. Only through a study of the individual thing is it possible to get an insight into the nature of things: only by an examination of the objects studied is real interest and a real desire for knowledge developed, only by this means is a real training possible. Whence it follows that he only is a good elementary teacher, who is not satisfied till the subject is known objectively. Such a teacher needs no warning against dealing with abstractions, for he has trained himself to think concretely for the sake of directing the dawning intelligence of his pupils. This is the secret of a good elementary method" (*Diesterweg*).

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[1] The term method as used in this essay has been perhaps sufficiently defined in the foregoing discussion. But since there is much ambiguity in the use of the term it may be well to add a word here. Really there is but *one* teaching method — the method in harmony with the mind's natural processes and with the logical structure of the subject to be mastered. Of course such a method is an ideal. What we really have is only an approximation — still an approximation close enough to be our chief guide in planning special "methods." A special method is obtained by applying the general principles of method to the teaching of some particular subject. Such an application is what is meant whenever the method of teaching geometry is referred to in this essay.

It has been said that "the history of a science reveals the method of teaching it." The first geometry was of course a body of empirical knowledge, not to be called a science. The first men who cultivated the science of geometry could not have begun with propositions and strict deductive proofs. Geometrical knowledge, like all other knowledge in its early stages, began with the observation of real things and individual relations. Then followed a classification of empirical knowledge, and by induction the facts of experience were organized into a science. In this way the propositions and proofs of geometry were evolved. In this way every human being repeats the history of his race. In this way he has to acquire the knowledge which really enters into his intellectual life, which becomes a part of his mind, and is not simply attached to it. The principles which shape a man's conduct — the generalizations which consciously or unconsciously determine the way in which he sets about his daily tasks — have been evolved from his experience. Principles of conduct and rules of procedure in trade or business may have been "learned," to be sure. But they do not become real possessions until experience has verified them, has shown their efficacy in the individual instances of one's career. We are here dealing of course with a very well known law of mental activity — that intelligence always proceeds from thing to name and symbol, from facts to principles and rules. In conformity with this law then, the facts of geometry must enter the learner's mind through experience. He must see and feel material bodies, their surfaces, their bounding lines, their corners; he must see that two vertical angles, that triangles under certain conditions are equal, by actually superposing them, he must see by cutting off the angles of a number of triangles and placing them side by side so that their vertices coincide, that the sum of the angles of a triangle is two right angles, he must see that a line is actually bisected or trisected by a given process by applying the process and testing his results, and soon. Under such circumstances no normal mind can be kept from attempting generalizations, from questioning the reason of the processes and results. But this is a great step forward. Once develop the questioning attitude, and then *satisfy* the inquiring mind and further progress and interest is assured. Now the questioning attitude has to come from an experience with the facts. If a boy wonders how the width of the marsh or the river has been measured without crossing it, or if he inquires how the height of the mountain or the tree has been measured without cutting down the one and climbing the other, he is in a proper condition to learn the geometrical principles underlying these measurements. Without the objects to be measured, and without performing the computations the measurement involves it would be difficult, perhaps impossible, to get any voluntary attention, any patient examination of the problem for the sake of the principles underlying its solution. The attitude of a boy who has measured heights and distances by using the propositions concerning the equality of triangles, toward those propositions themselves, is very different from the attitude of the boy whose first experience with those propositions is drawn from the text-book or from a formal presentation by the teacher. The formal statement of the propositions and their logical proofs are to be introduced gradually and after the facts have been presented empirically.

It may be necessary, at this point, to answer an objection. It may be said that we do not need the objects themselves beyond the earliest stages, that vivid and accurate descriptions and diagrams will answer the purpose. To answer this objection we may consider how unsatisfactory to one knowing nothing about baseball would be a description of the game in the absence of bat, ball or playground — using merely verbal descriptions and diagrams. The inadequacy, in spite of endless talk, of such descriptions is obvious. Children know this so well that in teaching each other their games, they are accustomed to say, "Just do as I tell you, pretty soon you'll understand why." They realize the fact that acquaintance with the things and processes is a prerequisite to a proper understanding of the rules (principles) of the game.

Within the sphere of instruction think of the difficulty even mature pupils experience in grasping such concepts as "the measure of a dihedral angle," "sections of two pyramids equidistant from the vertices," "symmetrical trihedrals" and "symmetrical spherical triangles," without actual models. Younger pupils have of course similar difficulties in grasping the concepts and conditions with which *they* have to deal. I have already said that obstructed mental vision is painful and harassing just as obstructed physical vision is disagreeable and unsatisfactory in the highest degree. Clear mental perception can only follow clear physical perception, and for this the objects themselves must be at hand. Any description is necessarily made up largely of representative knowledge and hence inevitably fainter than the knowledge consisting chiefly of presentative elements. No verbal description can possibly delineate an object or a process so as to impress the mind with the vividness of a present experience.

This argument deals with such a commonplace that I should not dwell upon it, did I not know that there are still persons, who, from an exaggerated notion of the power of words, fail to realize the importance of much concrete illustration. So I do not hesitate to insert here a few more illustrations. If a pupil takes a paper parallelogram, cuts it in two along the diagonal, and superposes the two triangles thus obtained, the formal demonstration of the equality of these triangles easily grows out of the fact which he perceives in the individual instance. If a pupil passes a string around the circumferences of several different wheels, and then measures the string and also the diameter of each wheel, he can be led to see empirically that the circumference of a circle is about three and one-seventh of the diameter. Whence given the diameter, he may easily discover how to find the circumference of a tree, say, or of a water pipe, or round tank, and conversely, if he measures the circumference of any of these bodies he will know how to find the diameter. A funnel dipped into colored water and held in various positions brings out the conic sections. If a frame is made consisting of four wires joined at the ends so that the joints move easily, by manipulating this frame the pupil sees that a quadrilateral is not determined by its four sides; but if a similar frame is made of three wires, he sees that three lines determine only one triangle, and hence that all triangles having the three sides of the one equal to the three sides of the other each to each are equal. In early instruction then the presentation of a new subject, and of each new phase of a subject, must supply the experiences, — the empirical knowledge, which forms the basis of subsequent generalizations, and logical proofs. Of course mere passive recipiency to experiences and sensations will not do. The pupil's mind must be alert. He must become *productively active*. That is to say, the subject, the individual thing must unfold before him through his own exertion under the stimulating lead of his teacher.

2. Here we enter upon the second special direction for successful instruction in geometry, viz., *the teacher and pupil must work together in discovering the facts and their relations.* This condition requires the pupil to express himself fully upon every thing observed, every relation discovered. Accurate drawing or construction shows that the learner's observation has been full and adequate. Such means of expression ought to be employed whenever practicable. But above all clear telling can always be insisted upon to give evidence of adequate and accurate observation. Frequently it gives the teacher the only means of knowing the actual state of the learner's knowledge. Errors of observation and inference are thus easily corrected, and the consequences of simulated attention and interest easily detected.

Here let me emphasize the great importance of absolute accuracy in the use of language. The time is coming when the pupil's data and his reasoning will be expressed almost entirely by words. Slovenly habits of expression result from or tend to produce muddled thought. Moreover the beauty and precision of mathematical (geometrical) reasoning will never be appreciated without insisting upon the utmost accuracy in the statement of every detail. This does not mean that each pupil must employ exactly the same words to convey an idea, but it means that *the idea, the whole of it, and nothing but the idea, shall be conveyed by each pupil.* If a pupil says "The angles of a triangle are equal to two right angles," "The sum of the angles made by one line meeting another is two right angles," he either has a wrong idea, or else the true idea is not clearly defined. In either case the difficulty could not be detected without requiring the pupil to express himself. But the pupil will not express himself fully and accurately upon what he observes and thinks unless the teacher calls upon him to do so. The teacher must therefore question the pupil. Every question is a challenge to attention and to the learner's power of expression. To quote a pregnant saying, the teacher must question the subject into his pupils, and then he must question it out of them. This is the most productive method of teaching geometry. A method of continuous exposition by the teacher or text-book is not a good method to employ.

This process of unfolding the subject step by step through the teacher's questions and the pupil's answers is the *method of development*, or the (falsely) so-called *method of discovery*. To be successful in the use of it the teacher must of course be master of the subject. He must be master of the subject in such a way that he can through his questions cause the pupil to reconstruct the science of geometry for himself. The teacher therefore must know the subject from the teacher's point of view, i. e. he knows it in all its details, and knows the relation of all these details to each other and knows how to bring them out in their logical dependence. Finally he must know how to cause the pupil to seize the relations of all these details and thus to feel that in each lesson he has conquered a definite portion of the subject. All this is to be done by much questioning, little telling. The pupil's response must come both in words (oral and written) and in measurements and constructions. A final test after the lesson has been fully developed in the manner indicated must call forth, from the pupil, a complete account of what has been gone over in the lesson.

Of course random questioning can never result in the real acquisition of knowledge, and the development of interest desired. The questions must be *terse, pointed, searching, perfectly intelligible to the pupil, and must follow each other in logical order.* Simply to question a subject into shreds, so to speak, a bewildering cross-examination of the subject to be learned, is worse than useless. A well-matured, well-constructed plan of questioning must be followed for each lesson. Pupils are quick to detect the power of a sure and steady guide through the intricacies of a subject, and after the first two or three trials willingly and gratefully submit themselves to such a leader.

Now no subject lends itself to the application of the method here advocated so readily and with such good effect as geometry. Beginning with the concrete the pupil is led step by step to the abstract. It compels him to find and observe carefully each detail and then to classify and arrange his discoveries so as to develop definite general notions. It thus compels the pupil to employ not only his physical but also his mental vision — not merely sight but insight as well. He does not simply know something about geometrical concepts, he understands them.

Undoubtedly progress through the subject will be more rapid by some other method, by the lecture method, for example, or by allowing definitions and demonstrations to be learned from dictation, or reproduced verbatim from a book. By one of the latter methods the pupil need not seek the truth, he receives it already found and ready made. Now to present the "net product of an inquiry without the inquiry which led to it" is to cultivate a reliance upon the verbal memory to the neglect of the power of overcoming difficulties and of assimilating experiences; moreover the accumulation of such unrelated mental stores is merely transitory — they are soon forgotten; there is no permanent gain of either knowledge or power. A method of continuous exposition is productive only with minds already developed, not with those *to be developed.* By its exclusive use with minds at all degrees of maturity the best results can never follow. Self-activity, interest, self-reliance, the power to be useful, these will never follow a method of instruction by which mental stores are imparted as so many free gifts. Fortunately such gifts are really impossible; there is no inheritance of knowledge and power. Only capabilities are inherited. There is but one universal inheritance — *ignorance;* one universal means provided by nature of rising above this inheritance — *self-exertion.* Who does not employ the means Nature has provided remains unlearned and helpless, though he may for a time simulate attainment and intellectual strength.

Evidently then if a teacher or the book tells the pupil that "Geometry is the science which treats of magnitude and form," "Magnitude is, etc.," the intellectual food is presented in a form in which it cannot be assimilated. No one can, of course, be sanguine enough to hope to make every one of his pupils an enthusiast by the best mode of presentation that could be devised. True it is, however, that unless interest is roused, and the mind is really trained through the knowledge acquired — if the pupil's powers are not developed and his susceptibility to culture is not quickened by it — it would be vastly better to exclude geometry from the schools. No teaching at all is better than a false instruction, yielding no real knowledge, and developing weakness instead of strength.

3. Grammar school instruction in geometry is to yield useful knowledge as well as discipline. This brings me to the third point to be considered, viz.: *early instruction in geometry must be practical.*

Now all instruction which provides abundant exercise for the application of what has been learned is practical in an important sense; and such applications must be adequately provided both in quality and quantity. Of course it will never do to be satisfied with *showing* the pupil the appropriate application. He must pass from knowledge *about* the thing to the knowledge *of* it that experience affords. It is not enough that the pupil sees and understands a given proposition: he must further be required to find its consequences and to solve problems depending upon it. He must continue the application by measuring with divider and rule, by drawing, cutting, superposing and constructing; and so verify in fresh instances what he has just learned. The pupil's ability to make such applications of his knowledge is the best test of his progress in power as well as knowledge.

But something more may be expected of appropriate applications of grammar school geometry. Distances, areas, and solids have to be measured in every pursuit. There is thus every opportunity to make the pupil's geometrical knowledge bear directly upon daily life. Through the use of the foot rule, tape or chain, with or without some form of goniometer, data are obtainable, which with drawing and computation furnish endless facilities for developing principles or for rendering significant what has been learned or what is subsequently to be studied in detail. A single hour spent in the field or in the shop, for the collection of data to be made the basis of work in the class-room, will illuminate many days of teaching and learning. Again, nothing can exemplify the value of class-room instruction like a practical application to construction in the shop, or measurement in the field. *For geometry as for geography "field work" is well-nigh indispensable.*

This field work more than anything else will make the pupil's geometrical training *significant.* The countless illustrations of the elementary facts and principles of geometry to be found in nature, and in the structures made by man — in buildings, bridges, and machines, and the numerous applications to practical affairs will develop an interest in the subject for those who do not find in it an intrinsic interest and will of course, strengthen the interest of those who are attracted by the subject itself.

To sum up: Geometry has been introduced into the curriculum of the grammar school because of its educational value — because it yields a peculiar and important kind of knowledge and a highly desirable mental discipline.

That this educational value may be realized, teachers are reminded of the following principles: —

1. Early instruction in geometry should be object-teaching.
2. An accurate record should be made and kept by the pupil of his observations and of the definitions or propositions which his examination of the object or objects has developed.
3. In all this work the pupil should be led to express himself by drawing, by construction, and in words, as fully and accurately as possible. The language finally accepted by the teacher should be the language of the science and not a temporary phraseology to be set aside later.
4. The pupil is to convince himself of geometrical truths primarily through measurement, drawing, cutting, superposition and construction, not by a logical demonstration, but gradually (especially during the last year of the work) the pupil should be led to attempt the general demonstration of all the simpler propositions.
5. The subject should be developed by the combined effort of the teacher and the pupil, i. e. the teacher and pupil are to coöperate

in reconstructing the subject for themselves. This is best accomplished by skilful questioning *without the use of a text-book containing definitions, solutions and demonstrations.* The teacher should follow a definite outline, which may be in the hands of the pupils: but, definitions, solutions, constructions, demonstrations and applications should be wrought out and preserved by the pupils. (It is assumed, of course, that the teacher has many books for personal inspiration and that he will, moreover, find in them many exercises which he can adapt to the needs of his classes.) The importance of frequent reviews must not be overlooked.

6. The subject-matter of each lesson should be considered in its relation to life, i. e. the actual occurrence in nature and in the structures or machines made by man of the geometrical forms studied, and the application of the propositions to the ordinary affairs of life should be the basis and the outcome of every exercise. That is, the pupil must not only acquire knowledge and skill but he must learn to apply them. To this end many examples for practice are necessary and field work and excursions to shops and factories are well nigh indispensable.
7. Accuracy and neatness are absolutely essential in all the work done by the pupils.

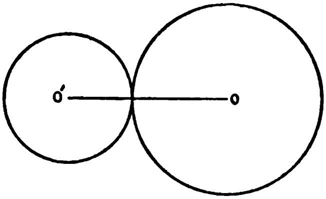
It will be observed that principles 1–5 deal with the acquisition of the *individual notion* and the transition to the *general notion* referred to in *the general principle of method* on page 9, and that the remaining principles provide for the proper application of what has been learned.

ILLUSTRATIVE CLASS EXERCISES.

In order to avoid needless repetition the illustrative class exercises which follow do not all exemplify all the principles above insisted upon. In the actual work of instruction it is expected that all those principles will be regarded in each exercise, so far as possible.

I.

The other day when we were looking at the printing press I called your attention to two wheels in this position (Teacher draws two circles free-hand, thus): —



Teacher. How would you describe two circles so situated? *Pupil.* The two circles just touch each other without intersecting. *T.* When two circles touch each other without intersecting we say that they are tangent (writing the word on blackboard) to each other. *T.*

What are tangent circles? *P.* Two circles are tangent to each other when they just touch each other (when they touch without intersecting each other). *T.* Three of you may go to the board and write the definition of tangent circles that you have just given. The others may write it in their note-books. Those who are at the blackboard may draw a pair of tangent circles. The class may observe. I shall ask another question presently. Did you know how to draw these circles so that they must be tangent? [Various answers, some of which may be correct but clumsily expressed.] *T.* Some of you are right but we must have a better statement for it. Call the first circle O and the second O′. Now join OO′. OO′ is called ***the line of centers*** of the two circles. What is the line of centers of two (tangent) circles? *P.* The line joining the centers of two (tangent) circles is called the line of centers. *T.* Write that on the board. The rest may put it in their note-books. Now compare the length of the line of centres with the two radii and what do you observe? *P.* The line of centers is equal to the sum of the radii. *T.* That is a part of the statement we have been seeking. Now when the line of centers is equal to the sum of the radii, what must be true of the position of the circles? *P.* When the line of centers of two circles equals the sum of the radii the two circles are tangent externally. *T.* That is the statement we have been seeking; you may write it on the board and in your note-books. *T.* Can you think of any use to be made of this statement or ***proposition?*** Some of the pupils may see that it serves to inform the machinist where the centers of two wheels are to be placed in order that the wheels may just touch (friction wheels). Otherwise the teacher brings out this fact by questioning, or partially brings it out and sends the pupils to the machine-shop to get full information.

Who can give examples of tangent circles beside those which we saw in the printing press? Each of you find an illustration, if possible, to put in your note-books before next time.

Draw two circles tangent externally, of which the radii are one inch and two inches respectively.

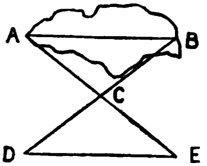
Draw five circles, each having a radius of one inch, and all tangent to a circle having a radius of two inches. Have you ever seen circles arranged in this way? Try to find examples: examine church windows. Can you draw a continuous line passing through the centers of all these outside circles? Draw and describe that line in your note-books.

NOTE. — Of course the case of two circles tangent internally would now be treated in a similar way. So also two circles wholly external, two intersecting circles, and two circles one of which lies wholly within the other.

II.

1. *T.* Last week we went out to measure the length of —— pond without rowing across it. You remember the lines we measured and the rough sketch of our measurements that we made at the time, especially you remember that the line that gave us the length of the

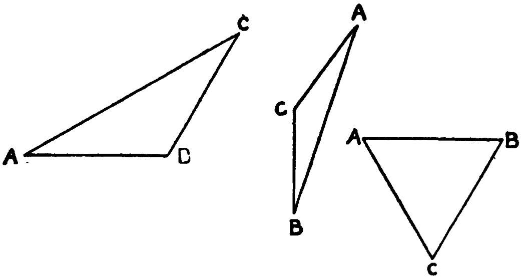
pond was quite a distance away from the pond. (To a pupil.) You may go to the board and make a sketch of our measurements. Pupil puts this sketch on the board.



T. Now what is the length to be measured? *P.* AB. *T.* What line did we measure? *P.* DE because you told us that DE was equal to AB. *T.* You may describe just what we did to get the line DE. *P.* You sent two boys to the two ends of the pond, where they placed two stakes A and B. Then one boy selected a point down the road at C from which the two stakes A and B could be seen. Then we measured the lines AC and BC. Then we extended the line AC beyond C until CE was equal to AC and placed a stake at E. We also extended BC beyond C until CD was equal to BC, and placed a stake at D. We then measured DE and found it to be 562 feet; you told us that DE was equal to AB, the length of the pond, and that we could prove it later. *T.* That we will do now. Look at the two triangles ABC and CDE and tell me what parts of one were made equal to certain parts of the other. *P.* AC equals CE, CD equals CB. *T.* Are there any equal angles? *P.* The angle ACB equals the angle DCE because they are opposite angles. [The equality of opposite angles has been shown before.] *T.* Then, in a single statement, what parts of the triangle ACB are equal to corresponding parts of the triangle DCE? Draw two such triangles on your papers. Letter them, putting your letters inside the angles. (Query: Is it necessary to have the triangles in just the position in which we have them on the blackboard?) Cut these two triangles out. Place them on each other so that the vertex C of ABC falls on the corresponding vertex C of DCE, and so that AC falls on and coincides with its equal CE. Then observe what happens. *P.* The two triangles coincide throughout. *T.* Show me just what you mean. (Pupil points out that each angle of ABC coincides with each angle of DCE and that each side of ABC coincides with each side of DCE.) *T.* Two pupils may go to the board and the rest may take their papers. What parts of one of these triangles are equal to corresponding parts of the other? Tell me *in words*, without naming the sides and the angle. Teacher finally obtains the statement from pupil: Two sides and the included angle of one triangle are equal to two sides and the included angle of the other. *T.* What did we then find to be true? *P.* All the parts of one triangle equal to all the parts of the other, or the triangles are equal throughout. *T.* Put that into a single statement. Teacher obtains from *P.*: When two sides and the included angle of one triangle are equal respectively to two sides, etc. *T.* We can show that this must be true always, as follows. The general proof of the proposition shoul dnow be given in the usual way, and need not be inserted here.

2. In the foregoing illustration the work was begun in the field. That may not always be practicable. In that case it is always possible to begin with drawing, as follows: Two pupils may go to the blackboard; the

rest may take their papers. Each pupil may draw any convenient triangle ABC thus: —

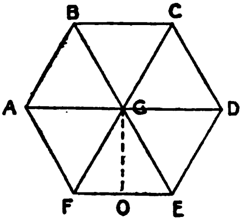


Now at a little distance draw A′B′ equal to AB. At A′ construct an angle equal to A, and at B′ and on the same side of A′B′, an angle equal to B. Produce the sides of these angles until they meet at C′. Cut the triangles out and apply A′B′C′ to ABC so that A′B′ falls on AB; then observe what happens, etc., in the same manner as before.

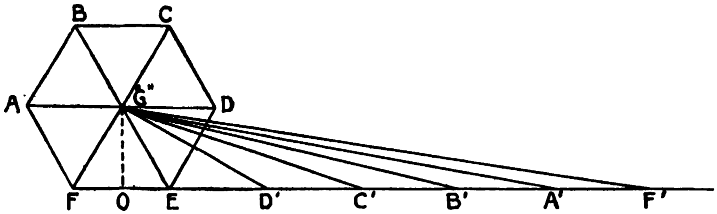
III.

1. *T.* It is often necessary to measure the areas of regular polygons, as, for example, regular pentagons or hexagons that form the floors of bow windows or other projections of buildings. Can any one give examples of such polygons? Such areas are very easily measured as follows: Two pupils may go to the blackboard. You may all draw a regular hexagon and find its center. (This construction is made possible by previous work.)

Connect the center with the vertices. Draw the apothegm and letter it GO. How do the triangles into which the hexagon is divided compare with each other? What is the area of each? Give me a statement for the area of the polygon. By questions *T.* finally obtains the statement that the area of the polygon is six times one side multiplied by the apothegm, divided by two. *T.* This is a correct statement, but we can get another which we shall want to use by and by in proving another proposition.



Produce FE to D′ making ED′ equal to ED, then produce FD′ to C′ making D′C′ equal to DC, and so on until we have produced FE to FF′ making A′F′ equal to AF thus: —



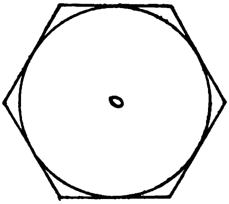
Now compare the area of the triangle FGE with each of the triangles EGD′, D′GC′, etc. *P.* These areas are all equal. *T.* Now compare the area of the triangle FGF′ and the polygon ABCDEF. *P.* The triangle and the polygon are equal in area. *T.* Why? Compare the base of the triangle FGF′ and the perimeter of the polygon: The altitude of the triangle and the apothegm of the polygon.

T. Can you give me a statement for the area of the hexagon in terms of the perimeter and apothegm? Teacher finally obtains the statement — the area of a regular hexagon is equal to half the product of its perimeter and apothegm, and thence that the area of any regular polygon equals, etc.

IV.

The Area of the Circle.

We want to find out how much tin is required to make the bottom of this circular dish, which is just six inches in diameter. In other words we want to know how to find the area of a circle. This you may discover as follows, but it will take us some time, probably several days. But every step of the process is so interesting that I am sure you will not find the whole process too long.



1. In the first place you may inscribe a circle *O* in a regular hexagon—make a good large hexagon. Now bisect the arcs intercepted by the sides of the hexagon and draw tangents at the points of division so as to obtain a regular circumscribed dodecagon. Measure the circumference of O and the perimeter of hexagon and dodecagon respectively. Which of them is nearer the length of the circumference? Bisect the arcs intercepted by the sides of the dodecagon and obtain a regular polygon of 24 sides. Compare the perimeter of this polygon with the circum-

ference. Does it differ more or less from the circumference than the foregoing perimeters differed from it? *T.* Suppose we continue this process, what length (line) does the perimeter of the polygon approach? *P.* It becomes more and more nearly equal to the circumference of the circle, and when the number of sides of the circumscribed polygon is very large the perimeter is no longer distinguishable from the circumference. *T.* All this time the area of the inscribed polygon has been changing too. What value has it been approaching? *P.* The area of the circle. *T.* It is usual to express this by saying — if the number of sides of a circumscribed regular polygon be indefinitely increased the perimeter of the polygon approaches more and more nearly to the circumference of the circle and the area of the polygon to the area of the circle.

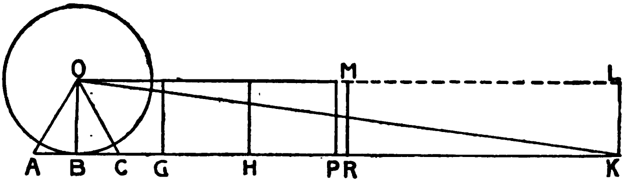
Now draw the apothegms of the three circumscribed polygons you have drawn. *T.* As the number of sides of the circumscribed polygons is increased does the apothegm change? *P.* No. It is equal to the radius of the circle.

Now we have shown heretofore that the *Area of a regular polygon equals perimeter times apothegm divided by* 2: and of course this rule for the area of a regular polygon is true, no matter how many sides the polygon has, i. e., no matter how near its perimeter approaches the circumference of the circle, and its area the area of the circle. In fact you will be able to prove, by and by, that we may actually substitute in the formula for the area of a regular polygon the words *circumference* for *perimeter*, *radius* for *apothegm*, *area of circle* for *area of* *polygon*, and the resulting statement will be true. If that is so what statement gives you the area of a circle. *P*. The area of a circle equals the circumference multiplied by the radius divided by two. *T*. If A stands for area, C for circumference and R for radius, we may write briefly $A = \frac{CR.}{2}$

Now we have already learned that $C = 2\pi R$. Putting $2\pi R$ in the place of C in the formula for the area and what do you get? *P*. $A = \frac{2\pi RR}{2} = \pi R^2$. *T*. Now translate your last result into words. *P*. The area of a circle equals $3\frac{1}{7}$ (accurately π) times the square of the radius.

T. Now we can return to our question about the amount of tin required to make the bottom of a circular dish, etc.

2. *T*. You may also see how the area of a circle equals (about) $3\frac{1}{7}$ times the square of the radius in this way. You may all draw the circle O at one end of your papers with a radius of one inch. Now draw the radius OB and draw the triangle OAC and suppose it to be one of the equal portions of a circumscribed regular polygon. Produce BC to K, i. e., until BK equals in length the perimeter of the circumscribed polygon, whatever that may be. Now join OK, and compare



the area of the triangle OBK with the area of the regular circumscribed polygon. *P.* These areas are equal. *T.* As the number of sides of the polygon is increased does BK grow shorter or longer? *P.* Shorter. *T.* What line does BK approach in length? *P.* The circumference of the circle. *T.* And what area does the area of the polygon approach? *P.* The area of the circle. *T.* But the area of the polygon, you said, equals the area of the triangle OBK. So the area of the triangle approaches the area of the circle as BK approaches the circumference of the circle. Now make the line BK just equal in length to the circumference of the circle, i. e., to $6\frac{2}{7}$ times the radius OB; then, as you will be able to prove by and by, the area of the triangle equals the area of the circle. Now complete the rectangle OBKL, of which the triangle OBK is the lower half. Now bisect the base BK by the perpendicular MR. *T.* The line BR is how many times the radius? *P.* $3\frac{1}{7}$. *T.* Compare the area of the triangle OBK and the rectangle OBRM. *P.* These areas are equal. *T.* Lay off the radius on BR as many times as possible, calling the points of division G, H, P, and erect perpendiculars at each of these points. You have divided the rectangle OBRM into three squares and a little rectangle, what is the value of each? *P.* Each square is the square of the radius, and the rectangle is $\frac{1}{7}$ the square of the radius, so that the rectangle OBRM is $3\frac{1}{7}$ times the square of the radius. But the area of the rectangle OBRM equals the area of the triangle OBK, and the area of the triangle of OBK equals the area of the circle O. Therefore the area of the circle O equals $3\frac{1}{7}$ times the square of the radius, or (accurately) $A = \pi R^2$.

OUTLINE OF WORK IN GEOMETRY

FOR THE LAST THREE YEARS OF THE GRAMMAR SCHOOL.

It is assumed that the work outlined below will require two recitation periods per week for the first two years, and one recitation period per week for the last year. A recitation period should not be less than half an hour in length. *It is also assumed that the work will be carried on in accordance with the principles stated on pages* 23–25.

Apparatus. Each pupil needs a pencil, a note-book (letter size), divided scale, small right triangle, compasses and protractor. The school should have a chain or steel tape for measuring distances in the field, a yard measure and a meter measure for measuring lengths indoors, and all the other apparatus needed to illustrate the work planned in the outlines which follow. Each school system or district should also have a surveyor's compass for measuring angles in the field. A very simple goniometer could be made by skilful pupils by mounting on a tripod a circular graduated board about a foot in diameter, having a movable arm turning on a pivot at the centre of the board. By sighting along this movable arm the direction of a given line is determined, then after moving the arm until the line of sight falls in the other side of the angle to be measured, this angle can, of course, be read off on the graduated circular board. [A very neat instrument of this description was made by one of the pupils of the Manual Training School of this city (Cambridge, Mass.).]

INTRODUCTION.

1. What we study in Geometry. (*The form and size of bodies and of their boundaries and edges.*)

2. *Space.* Bounded portion of space is a geometrical body. A geometrical body has only two properties, viz.: *shape* or *form*, and *size* or *magnitude*. Dimensions of a body. Cube, Prism, Pyramid, Cylinder, Sphere. Occurrence of each in nature and in the structures made by man. Geometrical bodies are called **solids**.

3. *Surfaces. Boundaries of solids.* 1. Plane or flat. 2. Curved. Dimensions of a surface.

4. *Lines.* 1. Edges of solids. 2. Boundaries of surfaces. 3. Intersections of surfaces. Lines are straight or curved. Dimensions of a line.

5. *Points.* 1. Corners of solids. 2. Ends of lines. 3. Intersections of lines. 4. A point designates a place. Representation to the eye of point, line and surface.

PART I. PLANE GEOMETRY.

We shall now take the surfaces, lines and points away from the solids to which they belong, and put them into (or on) a single plane where we may study their properties without considering the solids from which they were obtained.

Pencil point marks a point on paper. Moving point traces a line. Line moving not in direction of its length traces surface.

I. LINES.

1. *Kinds of lines.* 1. Straight, test of straightness. 2. Curved. 3. Broken. 4. Mixed or composite. Drawing and reading a line. Two points determine the position of a line.

A. STRAIGHT LINES.

2. *Shortest distance between two points. How many straight lines can be drawn between two points?*

3. *Directions of a straight line* (vertical, horizontal, oblique).

4. *Position of straight lines with respect to each other.* 1. Same direction. 2. *Parallel lines* also said to have the same direction. Through a given point to draw a line parallel to a given line (with triangle and ruler). 3. Lines having different directions are *oblique* to each other or *perpendicular* to each other.

B. CURVED LINES.

Recall definition of curved line. Commonest curve is *circumference of circle.* Occurrence in nature and in structures made by man. How to draw circle. *Center, Radius, Diameter, Arc, Chord, Tangent. Concentric Circles. Tangent and Intersecting Circles.*

Relation of Diameter to Circumference (to be determined empirically). Pupils measure both the diameter and the circumference in each case, and are led to see that circumference is approximately $3\frac{1}{7}$ times diameter.

To bisect a straight line.

To divide a straight line into any number of equal parts.

Measurement of straight lines. Units of length. Description and use of surveyor's chain and tape.

Drawing of measured lines. Drawing to scale. Scale of reduction. Reducing factor.

II. ANGLES.

1. *Definition. Terms needed. Reading an angle.*

2. *Magnitude and kinds. A degree.*

3. *The protractor* and its uses — to measure angles; to lay off angles.

4. *Equal angles. Construction* of an angle equal to a given angle. Sum and difference of angles.

5. *Groups of angles*: (a) About a common vertex — 1, adjacent; 2, supplementary adjacent; 3, sum of all the angles about a point. (b) Opposite angles. (c) Angles made by transversal and parallels — 1, exterior-interior; 2, alternate-interior; 3, sum of interior non-adjacent. Applications of converse of 1, 2, 3, of (c).

III. RECTILINEAR FIGURES.

A. TRIANGLES.

1. *Occurrence. Definition. Terms used* (sides, base, altitude, vertex).
2. *Kinds* (scalene, isosceles, equilateral, equiangular, acute, obtuse, right).
3. *Important properties.*

Sum of two sides compared with third side.

Sum of angles of a triangle. Corollaries. (What is a corollary?) Exterior angle.

Triangles or other figures may be *equal* or *equivalent* or *similar.*

Equal triangles; 1, if *two sides and included angle* of one triangle; or, 2, if *a side and two angles* of one triangle; or, 3, if *the three sides* of one triangle are equal to the three sides of another triangle. Applications in each case.

Angles at the base of an isosceles triangle. Corollaries.

Converse of last proposition. (What is a proposition?)

Position of vertex of isosceles triangle with respect to middle point of the base.

Two isosceles triangles having a common base. Constructions: 1. To bisect a given line. 2. To bisect a given angle. 3. From a given point to let fall a perpendicular upon a given line.

Relation of sides to opposite angles in any triangle, and conversely.

Only one perpendicular can be drawn from a point to a line. The perpendicular is the shortest distance from a point to a line and measures that distance.

B. QUADRILATERALS AND OTHER POLYGONS, AND THEIR MOST IMPORTANT PROPERTIES.

1. *Occurrence and kinds of quadrilaterals. Terms used.*

2. *Important properties common to all parallelograms.*

Sum of any two consecutive angles.

Diagonal divides parallelogram into two equal triangles.

Opposite sides are equal.

Opposite angles are equal.

Diagonals bisect each other.

If the opposite sides of a quadrilateral are equal, the figure is parallelogram.

3. *The Trapezoid.*

Construct trapezoids of different shapes.

Angles in every trapezoid.

Sum of two angles adjacent to a non-parallel side.

In isosceles trapezoid angles adjacent to same parallel are equal.

Diagonal divides trapezoid into two unequal triangles.

Median Line one-half the sum of bases.

4. *Polygons.*

In general. Regular Polygon. Diagonals: Number of different diagonals.

Sum of interior angles of a polygon.

Center of regular polygon. Angle at center.

Construction of regular polygons.

Using protractor and angle of polygon.

Using angle at center.

Without protractor — consider only regular hexagon.

C. AREAS.

1. *Units of Area.* Sq. ft., sq. in., Qm., Qdm., Qcm.

2. (a) *How to measure — find the area of* — a square. Obtain the formula $A = S^2$ (S stands for the length of one side). (b) How to find the side of a square when the area is given. Obtain formula $S = \sqrt{A}$. (What is a formula?)

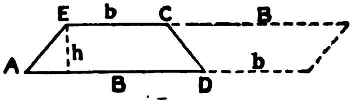
3. (a) How to find the area of a rectangle. Obtain formula $A = a\,b$. (b) How to find base when altitude and one dimension are given (known). Obtain formulæ $b = \frac{A}{a}$: $a = \frac{A}{b}$.

4. (a) Area of a parallelogram. Obtain formula $A = a\,h$. (h = altitude.) (b) From the formula $A = a\,h$ obtain $a = \frac{A}{h}$ and $h = \frac{A}{a}$ and translate these results into words.

5. (a) Area of a triangle. Obtain formula $A = \frac{b\,h}{2}$. (b) From this formula obtain $h = \frac{2\,A}{b}$ and $b = \frac{2\,A}{h}$, and translate the results into words.

6. Comparison of different triangles having equal bases and equal altitudes.

7. The area of a trapezoid. Obtain formula $A = \frac{B+b}{2} \times h$, or $\frac{h}{2}(B+b)$.



8. Area of any polygon.

9. Area of a regular polygon. Obtain formula $A = \frac{ph}{2} n$, in which p stands for one side, h for the apothegm, and n for the number of sides. Also obtain $A = \frac{Ph}{2}$ in which P is perimeter of polygon.

Simple Exercises in the Transformation and Subdivision (Partition) of Rectilinear Figures.

(a) Exercises in Transformation, i. e., Changes in form but not in area.

1. To transform a parallelogram into an equivalent rectangle.

2. To transform a parallelogram into an equivalent triangle.

3. To transform a scalene triangle into an equivalent isosceles triangle.

4. To transform a triangle into an equivalent parallelogram.

5. To transform a trapezoid into an equivalent triangle.

6. To transform a trapezoid into an equivalent parallelogram.

7. To transform a regular pentagon into an equivalent triangle.

(b) Exercises in the subdivision (partition) of areas. Here recall the problem of dividing a line into equal parts, the construction for which may now be proved.

1. To divide a triangle into five (any number of) equivalent parts by lines drawn from the vertex.

2. To divide a parallelogram into six (any number of) equivalent parts by lines drawn parallel to one side.

D. THE PYTHAGOREAN PROPOSITION.

(a) In the case of the isosceles right triangle. (b) In the case of the scalene right triangle whose sides are to each other as 3 : 4. (c) In general. (d) Applications.

1. Construct a square equivalent (a) to the sum of two given squares; (b) to the difference of two given squares.

2. Construct squares equivalent to two, three, four, etc., times a given square.

3. Given the sides of a right triangle to find the hypothenuse.

4. Given the hypothenuse and one side to find the other side.

5. Find the area of an equilateral triangle.

E. SIMILAR FIGURES — *more especially* SIMILAR TRIANGLES.

1. Similar figures. Show by comparison and measurements:

(a) That two similar figures have the same number of sides;

(b) In two similar figures each angle of the one has its corresponding *equal* angle in the other;

(c) In two similar figures, corresponding (homologous) sides have a constant ratio; — or

Two figures are similar if the angles of one are respectively equal to the angles of the other and if the homologous sides of one have the same ratio as the homologous sides of the other.

2. Two triangles are similar if two angles of one are equal respectively to two angles of the other.

F. HOW THE AREAS OF SIMILAR FIGURES ARE RELATED TO EACH OTHER.

1. (The areas of) Two squares have the same ratio as (are to each other as) the squares of their sides.

2. The areas of two similar triangles have the same ratio (are to each other as) the squares of their homologous sides.

APPLICATIONS: —

1. To find the height of (*a*) a tree or (*b*) of a tower by means of its shadow.

2. To find the height of a tree or tower without using the shadow.

3. To find the height of a hill.

4. To measure an inaccessible distance in the field.

IV. CURVILINEAR FIGURES.

A. THE CIRCLE.

1. Definitions. Review properties of the circle previously considered.

2. *Angle at the center* and *inscribed angle.*

3. Sector. Quadrant. Segment.

4. *The situation of the center of the circle with respect to the middle point of any chord.*

APPLICATIONS: —

   (a) To find the center of a circle or of an arc.

   (b) To draw a circle through any three given points not in the same straight line.

2. The distances of equal chords from the center of the circle.

3. The distances of unequal chords from the center of the circle.

4. Equal chords correspond to equal angles at the center.

5. The chord of 60° compared with the radius.

APPLICATIONS: —

(a) To inscribe a regular hexagon in a circle.

(b) To inscribe a regular triangle in a circle.

6. An inscribed angle is measured by half the intercepted arc. Angle inscribed in a semi-circle.

7. Angles inscribed in the same segment.

8. An angle inscribed in a semi-circle.

9. Position of the center with respect to the point of contact of a tangent; or, the tangent is perpendicular to a radius at the point of contact.

APPLICATIONS: —

(*a*) At a given point on the circumference draw a tangent to the circle.

(*b*) From a given point without the circle draw a tangent to the circle. (How many such tangents may be drawn from the same point?)

(*c*) To draw a circle tangent to the sides of any given angle.

(*d*) To inscribe a circle in a triangle.

10. How to draw regular polygons:

(*a*) With the protractor and circle.

(*b*) Without a protractor only certain regular polygons can be inscribed in a circle: (1) Inscribed regular square, octagon, polygon of sixteen sides, etc., (2) Regular hexagon, triangle, dodecagon, polygon of twenty-four sides, etc., (3) Other inscribed regular polygons.

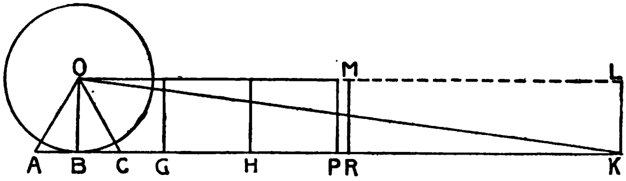
11. To circumscribe a regular polygon about a circle.

12. To inscribe a circle in a regular polygon and to circumscribe a circle about a regular polygon.

THE AREA OF THE CIRCLE.

1. The area of the circle equals about 3.14 times the square of the radius; or accurately and briefly $A = \pi r^2$, in which A stands for the area of the circle, r for the radius, and π for the ratio of the circumference to the diameter.

This truth can be approximately seen (with much help from the teacher) by the pupils from the following figure, in which Bk is equal to the circumference of the circle. See Illustrative Class Exercises, pages 32–35.



2. From the formula $A = \pi r^2$ obtain the following: (a) $A = \pi r r = 2 \pi r \times \frac{r}{2} = \frac{Cr}{2}$ in which C stands for the circumference. (b) Since $A = \pi r^2$, $\frac{A}{\pi} = r^2$, or $r = \sqrt{\frac{A}{\pi}}$. (Pupils should translate these results into words.)

[3. Area of a circular ring; of a sector; of a segment.]

RELATION OF TWO CIRCUMFERENCES AND OF TWO CIRCLES TO EACH OTHER.

1. Two circumferences bear the same ratio to each other as their radii.

2. (The areas of) Two circles bear the same ratio to each other as the squares of their radii.

B. THE ELLIPSE.

1. Occurrence in nature and in structures made by man. A conic section. To draw an ellipse.

2. Properties. Compared with circle. Major Axis. Minor Axis. Eccentricity. Foci. A radius vector.

3. (a) The sum of the radii vectores for all points is constant, and equal to the *major axis*. (b) The radius vector drawn to either end of the minor axis is equal to half the major axis.

4. Constructions. (a) To draw an ellipse when the axes are given. (b) To draw an ellipse when one axis and the eccentricity are given.

[5. Area of the ellipse equals $3\frac{1}{7}$ times the rectangle of the semiaxes, or briefly and accurately, $A = \pi a b$. The circumference of an ellipse is approximately given by the formula $C = 2\pi\sqrt{\frac{1}{2}(a^2 + b^2)}$.]

PART II. SOLID GEOMETRY.

A. THE PLANE AND STRAIGHT LINE IN SPACE.

Recall the fact that in Geometry we study the *form* and *size* of bodies. Thus far we have studied only the form and size of the surfaces and lines which bound the solids, and we have studied these surfaces and lines by themselves — apart from the solids to which they belonged. We have taken the surfaces and lines away from the solids — so to speak — and after putting them into a single plane, we have studied their properties. We are now to learn something more of surfaces and lines in connection with the solids to which they belong, when they are not confined to a single plane, when they are *in space* as it is called. (Illustrations.)

We shall also study the properties of the solids themselves. (Illustrations.) That is, we are now to study "SOLID GEOMETRY."

A surface moving not in the direction of one of its dimensions traces a solid.

1. A *plane* is *determined* (a) by three points, not in the same straight line; (b) by a line and some point not in the line; (c) by two intersecting straight lines; (d) by two parallel lines.

2. A line may be (1) perpendicular to a plane, (2) parallel to a plane, (3) make an oblique angle with a plane.

3. Dihedral angle. How measured.

4. Trihedral angle. Polyhedral angle.

B. THE CUBE.

1. Occurrence in nature and in structures made by man. Bounding surfaces. Edges. Vertices and trihedral angles. Axes. Pattern. *Definition.*

2. Solid measure (or cubic measure) or measure of *volume.* Units of solid measure, or how volumes are measured.

3. How to measure — find (1) the surface and (2) the volume of — a cube when one edge is given. Obtain $\begin{cases} S = 6e^2. \\ V = e^3. \end{cases}$

4. How to find an edge when the volume is given. Obtain $e = \sqrt[3]{V}$

C. PRISMS.

1. Occurrence in nature and in structures made by man. Important lines and surfaces: bases, faces, edges, altitude, vertices (of whole prism).

Definition. Kinds. (1) Right, oblique. (2) Parallelopiped. (3) Regular. (4) Triangular, quadrangular, hexagonal, polygonal. Pattern. Definition.

2. How to measure — find (1) the surface and (2) the volume of a right prism.

(1) Lateral area (surface). Total area (surface).

(2) The volume. (First show how the volume of a right parallelopiped is obtained. Then show that volume of any right prism is similarly obtained.)

(3) How may the base or the altitude be found from the volume, and either base or altitude as the case may be? Obtain formulæ.

D. THE CYLINDER.

1. Occurrence in nature and in the structures made by man. Comparison with prism. Bases. Elements. Altitude. Lateral surface. Pattern. Definition.

2. Section of cylinder. (1) Circle. (2) Ellipse.

3. How a cylinder may be generated.

4. How to measure — find (1) the surface and (2) the volume of a right cylinder.

(1) (a) Lateral area. (b) Total area.

(2) The volume. Compare with volume of prism and show that volume of cylinder is obtained in same way. Briefly $V = Bh = \pi r^2 h$.

(3) How may the base or altitude be found from the volume and either altitude or base, as the case may be?

5. The volume of a hollow right cylinder.

E. THE PYRAMID.

1. Occurrence in nature and in structures made by man. Base, faces, altitude, vertex, edges. Kinds, (1) Right. (2) Oblique. (3) Regular. (4) Triangular, quadrangular, polygonal. Pattern of the regular pyramid. Definition of a pyramid.

2. How to measure (find (1) the surface and (2) the volume of) a regular pyramid.

(1) (a) The lateral surface. (b) The total surface.

(2) Show with model or by cutting that the volume of a regular (triangular) pyramid is equal to $\frac{1}{3}$ the volume of a regular (triangular) prism having the same base and altitude. Briefly $V = \frac{1}{3} B h$.

Obtain from this (1) $B = \frac{3V}{h}$, and (2) $h = \frac{3V}{b}$, and translate the results into words.

F. THE CONE.

1. Occurrence in nature and in the structures made by man. Base. Convex surface. Altitude. Vertex. Elements. Definition. Kinds. (1) Right. (2) Oblique. How a right cone is generated. Conic sections — Circle, Ellipse, Parabola, Hyperbola, Triangle — Cut out of putty or clay, or a potato or turnip. Pattern of a right cone.

2. How to measure (find the (1) surface and (2) volume of) a right cone.

(1) Convex surface. Total area when any two of radius, altitude and element are known.

(2) The volume of a cone equals one-third the volume of a right cylinder having the same base and altitude. Show with model filling with water or sand.

Whence briefly, $V = \frac{B h}{3} = \frac{\pi r^2 h}{3}$ since $B = \pi r^2$.

G. FRUSTUM OF PYRAMID.

1. Occurrence: made by cutting off a pyramid by a plane parallel to the base. Important lines and surfaces on the frustum. Kinds. Pattern of frustum of regular pyramid.

2. How to measure (find (1) the area and (2) volume of) a frustum of a regular pyramid.

(1) The lateral area equals the sum of the trapezoids forming its faces. Total area.

(2) The volume equals the difference between the original pyramid and the pyramid cut off.

H. Frustum of Cone.

1. Occurrence: made by cutting off a cone by a plane parallel to the base. Definition. Important lines and surfaces on the frustum. Kinds. Pattern of frustum of right cone.

2. How to measure (find the (1) surface and (2) volume of) the frustum of a right cone.

(1) The lateral area (convex surface). Obtain the formula $A = (R + r) \pi s$.

(2.) The volume. Proceed as in the case of the frustum of pyramid.

I. The Sphere.

1. Occurrence in nature and in structures made by man. Definition. Important straight lines and points of the sphere (Diameter, Radius, Axis, Pole). Sections made by planes, and lines on the surface of the sphere (Small Circle, Great Circle, Equator, Parallels, Meridians, Quadrants). (d) How a sphere may be generated. Parts of the sphere (Hemisphere, Spherical Segment, Spherical Sector, Zone). Can you make a pattern of a sphere in a plane?

2. How to measure (find (1) the surface and (2) the volume of) a sphere.

(1) The surface (area). Wrap a sphere with narrow waxed tape, and also wrap its *enveloping* right cylinder. Now unwrap, and comparing amounts of tape, observe that area of sphere equals lateral area of enveloping cylinder. But lateral area of enveloping cylinder equals $2 \pi r \times r = 2 \pi r^2$. Translate this result into words; also observe that area of sphere also equals four times the area of a great circle of that sphere.

(2). Volume of the sphere. Fill the hollow sphere with water and then pour it into the enveloping cylinder. How full is the cylinder? Hence volume of sphere equals $\frac{2}{3}$ of volume of enveloping cylinder. But volume of cylinder equals $\pi r^2 \times 2r = 2\pi r^3$. Hence $V = \frac{2}{3}$ of $2\pi r^2 = \frac{4\pi r^2.}{3}$ Translate this result into words.

MEASUREMENT OF IRREGULAR BODIES.

The volume of an irregular solid may always be obtained by putting the solid into a vessel of known dimensions (usually a rectangular parallelopiped), and then filling the vessel with water or fine sand. If then the solid be taken out and the remaining volume of sand or water be measured, we have only to subtract this last volume from the volume of the vessel to obtain the volume of the solid.

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By E. A. BOWSER, Professor of Mathematics, Rutgers College, New Brunswick, N. J. Half leather. 402 pages. Price by mail, $1.40. Introduction price, $1.25. Plane Geometry. 250 pages. Introduction price, 75 cents.

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